Appendix A. Derivation of Formula, Fourier Series of an NRZ-L Data Stream

Formula:

An NRZ-L data stream of bit rate, **r**, and **c** bits in the pattern before repeating can be expressed as the following Fourier Series:

$$F_n = \frac{\sin\left(\frac{n\pi}{c}\right)}{n\pi} \sum_{k=1}^{c} R_k * \exp\left(\frac{-jn\pi(2k-1)}{c}\right)$$

where

n = # of harmonic

F_n = nth harmonic such that $f(t) = \sum_{n=-\infty}^{\infty} F_n * \exp(jn\omega_o t)$;

f(t) = the time domain representation of the data stream

$\omega_o = 2\pi f_o$, $f_o = \frac{r}{c}$

r = bit rate

c = # of values before repeating (frame length)

R_k = assigned variable, 1 or 0, in the kth position within the pattern or frame.

$a_n = 2\ \mathrm{Re}\ \{F_n\}$ (cosine term)

$b_n = -2\ \mathrm{Im}\ \{F_n\}$ (sine term), such that

$f(t) = a_o + \sum_{n=1}^{\infty} a_n * \cos(n\omega_o t) + \sum_{n=1}^{\infty} b_n * \sin(n\omega_o t)$.

Rewriting the expression for F_n, using Euler's identity:

$$F_n = \frac{\sin\left(\frac{n\pi}{c}\right)}{n\pi} * \sum_{k=1}^{c} R_k \left[\cos\left(\frac{n\pi(2k-1)}{c}\right) - j * \sin\left(\frac{n\pi(2k-1)}{c}\right)\right]$$

$$a_n = \frac{2 * \sin\left(\frac{n\pi}{c}\right)}{n\pi} \sum_{k=1}^{c} R_k * \cos\left(\frac{n\pi(2k-1)}{c}\right)$$

$$b_n = \frac{2 * \sin\left(\frac{n\pi}{c}\right)}{n\pi} \sum_{k=1}^{c} R_k * \sin\left(\frac{n\pi(2k-1)}{c}\right)$$

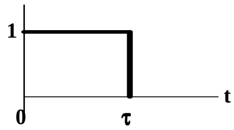
Derivation:

Definitions

$$\mathrm{rect}\left(\frac{t}{\tau}\right) \equiv \begin{Bmatrix} 1, |t| < \frac{\tau}{2} \\ 0, |t| > \frac{\tau}{2} \end{Bmatrix}$$, where τ is the width of the pulse.

$$\mathrm{rect}\left(\frac{t-\frac{\tau}{2}}{\tau}\right) \equiv u(t) - u(t-\tau)$$, where $u(\psi) \equiv \begin{Bmatrix} 1, \psi > 0 \\ 0, \psi < 0 \end{Bmatrix}$.

$\mathrm{rect}\left(\frac{t-\frac{\tau}{2}}{\tau}\right)$ is graphically represented by

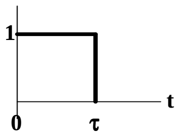


Proof

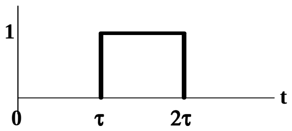
Let f(t) = the data stream function in the time domain.
F(ω) = Fourier transform of f(t), such that $f(t) \Leftrightarrow F(\omega)$

$$\mathrm{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau \mathrm{Sa}\left(\frac{\omega\tau}{2}\right), \text{ where } \mathrm{Sa}\left(\frac{\omega\tau}{2}\right) \equiv \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$$

Using the delay property of Fourier Transforms, $g(t - t_o) \Leftrightarrow e^{-j\omega t_o} G(\omega)$, t_o (the time delay), can be expressed as $t_o = \frac{(2k-1)\tau}{2}$. **k** is defined as a "bit position", or the time slot of a given data rate where the assigned bit will be either a "1" or "0". With **c** bit positions before the pattern repeats, **k** will take on a value, **k** = 1, 2, 3 ... **c**. **k** can be easily seen by the example below.



k=1, rect is displaced only by τ/2.



k = 2

In the example on the immediate left, k=2, so $t_o = 3\tau/2$. Defining **r** as the bit rate, $\boldsymbol{\tau = 1/r}$.

<u>Proof</u> *continued*

Substituting $t_o = \frac{(2k-1)\tau}{2}$ into the expression of the Fourier transform of a time delayed signal, $g(t - t_o) \Leftrightarrow e^{-j\omega t_o} G(\omega)$, and letting g(t) = $\mathrm{rect}\left(\frac{t}{\tau}\right)$, the Fourier transform pair of the **k**th bit position is:

$$\mathrm{rect}\left(\frac{t - \frac{(2k-1)\tau}{2}}{\tau}\right) \Leftrightarrow \exp\left(\frac{-j\omega(2k-1)\tau}{2}\right) * \tau * \mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$$

Using the property of superposition (linearity) of Fourier transforms, $a_1f_1(t) + a_2f_2(t) \Leftrightarrow a_1F_1(\omega) + a_2F_2(\omega)$, it is possible to use the Fourier transform of a bit position (shown above), and superimpose it with other bit positions. Since only bit positions that are a “1” should be included, the variable $\mathbf{R_k}$ is introduced, with a value of “1” or “0”, corresponding to the data stream at the **k**th position of the repeating pattern or frame.

The Fourier transform pair of the data stream of length **c**, can be expressed as: $\sum_{k=1}^{c} R_k * \mathrm{rect}\left(\frac{t - \frac{(2k-1)\tau}{2}}{\tau}\right) \Leftrightarrow \sum_{k=1}^{c} R_k * \exp\left(\frac{-j\omega(2k-1)\tau}{2}\right) * \tau * \mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$.(1)

Since the code is periodic, with period $\frac{c}{r} = T$, a Fourier series can be obtained from the transform using the property that $F_n = \frac{1}{T} * F(\omega)\Big|_{\omega = n\omega_o}$, (2) $\omega_o = \frac{2\pi}{T} = \frac{2\pi r}{c}$. In the above equation, **n** is the harmonic number of the series and $\mathbf{F_n}$ is the complex Fourier series representation for the harmonic such that: $f(t) = \sum_{n=-\infty}^{\infty} F_n * \exp(jn\omega_o t)$. **f(t)** is defined as the time domain signal consisting of integral multiples (>0) of the repeating data pattern. The above relationship is applicable since the signal has finite power and $f\left(\frac{\pm T}{2}\right) \approx 0$.[1]

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[1]Ferrel G. Stremler, *Introduction to Communication Systems*, Third ed. Reading, Massachusetts: Addison-Wesley, 1990. Basic Fourier transforms are also from this text.

Proof *continued*

Using the property of (2), and **F(ω)** from (1), the Fourier series of the data stream, **f(t)**, can be expressed as :

$$F_n = \frac{r}{c} * \sum_{k=1}^{c} R_k * \exp\left(\frac{-jn\left(\frac{2\pi r}{c}\right)(2k-1)\left(\frac{1}{r}\right)}{2}\right) * \frac{1}{r} * Sa\left(\frac{n\left(\frac{2\pi r}{c}\right)\frac{1}{r}}{2}\right)$$

$$F_n = \frac{r}{c} * \sum_{k=1}^{c} R_k * \exp\left(\frac{-jn2\pi r(2k-1)}{2rc}\right) * \frac{\sin\left(\frac{n\pi r}{rc}\right)}{\frac{n\pi r}{rc}} * \frac{1}{r}$$

$$F_n = \frac{r}{c} * \frac{1}{r} * \frac{rc}{n\pi r} * \sin\left(\frac{n\pi r}{rc}\right) * \sum_{k=1}^{c} R_k * \exp\left(\frac{-jn2\pi r(2k-1)}{2rc}\right)$$

$$F_n = \frac{\sin\left(\frac{n\pi}{c}\right)}{n\pi} \sum_{k=1}^{c} R_k * \exp\left(\frac{-jn\pi(2k-1)}{c}\right)$$

To express $\mathbf{F_n}$ in terms of real coefficients of sine and cosine, define $\mathbf{a_n}$ and $\mathbf{b_n}$ as:

$a_n = 2\ \mathrm{Re}\ \{F_n\}$ (cosine term)
$b_n = -2\ \mathrm{Im}\ \{F_n\}$ (sine term), such that

$$f(t) = a_o + \sum_{n=1}^{\infty} a_n * \cos(n\omega_o t) + \sum_{n=1}^{\infty} b_n * \sin(n\omega_o t).$$

Rewriting the expression for F_n, using Euler's identity:

$$F_n = \frac{\sin\left(\frac{n\pi}{c}\right)}{n\pi} * \sum_{k=1}^{c} R_k\left[\cos\left(\frac{n\pi(2k-1)}{c}\right) - j * \sin\left(\frac{n\pi(2k-1)}{c}\right)\right]$$

$$a_n = \frac{2 * \sin\left(\frac{n\pi}{c}\right)}{n\pi} \sum_{k=1}^{c} R_k * \cos\left(\frac{n\pi(2k-1)}{c}\right)$$

$$b_n = \frac{2 * \sin\left(\frac{n\pi}{c}\right)}{n\pi} \sum_{k=1}^{c} R_k * \sin\left(\frac{n\pi(2k-1)}{c}\right)$$