Appendix A. Derivation of Formula, Fourier Series of an NRZ-L Data Stream

Formula:

An NRZ-L data stream of bit rate, \( r \), and \( c \) bits in the pattern before repeating can be expressed as the following Fourier Series:

\[
F_n = \sin \left( \frac{n\pi}{c} \right) \sum_{k=1}^{\infty} R_k \exp \left( -jn\pi \left( \frac{2k-1}{c} \right) \right)
\]

where

- \( n = \# \) of harmonic
- \( F_n = \) nth harmonic such that \( f(t) = \sum_{n=-\infty}^{\infty} F_n \exp(jn\omega_0 t) \);
- \( f(t) = \) the time domain representation of the data stream
- \( \omega_0 = 2\pi f_o, f_o = \frac{r}{c} \)
- \( r = \) bit rate
- \( c = \# \) of values before repeating (frame length)
- \( R_k = \) assigned variable, 1 or 0, in the kth position within the pattern or frame.

\( a_n = 2 \text{ Re} \{F_n\} \) (cosine term)
\( b_n = -2 \text{ Im} \{F_n\} \) (sine term), such that

\[
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t).
\]

Rewriting the expression for \( F_n \), using Euler’s identity:

\[
F_n = \sin \left( \frac{n\pi}{c} \right) \* \sum_{k=1}^{\infty} R_k \left[ \cos \left( \frac{n\pi(2k-1)}{c} \right) - j \* \sin \left( \frac{n\pi(2k-1)}{c} \right) \right]
\]

\[
a_n = \frac{2 \* \sin \left( \frac{n\pi}{c} \right)}{n\pi} \sum_{k=1}^{\infty} R_k \* \cos \left( \frac{n\pi(2k-1)}{c} \right)
\]

\[
b_n = \frac{2 \* \sin \left( \frac{n\pi}{c} \right)}{n\pi} \sum_{k=1}^{\infty} R_k \* \sin \left( \frac{n\pi(2k-1)}{c} \right)
\]
Derivation:

Definitions

\[
\text{rect} \left( \frac{t}{\tau} \right) \equiv \begin{cases} 
1, & |t| < \frac{\tau}{2} \\
0, & |t| > \frac{\tau}{2} 
\end{cases}, \quad \text{where } \tau \text{ is the width of the pulse.}
\]

\[
\text{rect} \left( \frac{t - \frac{\tau}{2}}{\tau} \right) \equiv u(t) - u(t - \tau), \quad \text{where } u(\psi) \equiv \begin{cases} 
1, & \psi > 0 \\
0, & \psi < 0 \end{cases}.
\]

\[
\text{rect} \left( \frac{t - \frac{\tau}{2}}{\tau} \right) \text{ is graphically represented by}
\]

\[
\begin{array}{c}
0 \\
\tau \\
1 \\
t
\end{array}
\]

Proof

Let \( f(t) = \) the data stream function in the time domain.

\[ F(\omega) = \text{Fourier transform of } f(t), \quad \text{such that } f(t) \Leftrightarrow F(\omega) \]

\[
\text{rect} \left( \frac{t}{\tau} \right) \Leftrightarrow \tau \text{Sa} \left( \frac{\omega \tau}{2} \right), \quad \text{where } \text{Sa} \left( \frac{\omega \tau}{2} \right) \equiv \frac{\sin \left( \frac{\omega \tau}{2} \right)}{\frac{\omega \tau}{2}}
\]

Using the delay property of Fourier Transforms, \( g(t - t_o) \Leftrightarrow e^{-j\omega t_o} G(\omega) \), \( t_o \) (the time delay), can be expressed as \( t_o = \frac{(2k - 1)\tau}{2} \). \( k \) is defined as a “bit position”, or the time slot of a given data rate where the assigned bit will be either a “1” or “0”. With \( c \) bit positions before the pattern repeats, \( k \) will take on a value, \( k = 1, 2, 3 ... c \). \( k \) can be easily seen by the example below.

In the example on the immediate left, \( k = 2 \), so \( t_o = 3\tau/2 \). Defining \( r \) as the bit rate, \( \tau = 1/r \).

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Proof continued

Substituting \( t_0 = \frac{(2k-1)\tau}{2} \) into the expression of the Fourier transform of a time delayed signal, \( g(t - t_0) \leftrightarrow e^{-j\omega t_0} G(\omega) \), and letting \( g(t) = \text{rect}\left(\frac{t}{\tau}\right) \), the Fourier transform pair of the kth bit position is:

\[
\text{rect}\left(\frac{t - \frac{(2k-1)\tau}{2}}{\tau}\right) \leftrightarrow \exp\left( -j\omega \frac{(2k-1)\tau}{2} \right) * \tau * \text{Sa}\left(\frac{\omega \tau}{2}\right)
\]

Using the property of superposition (linearity) of Fourier transforms, \( a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) \), it is possible to use the Fourier transform of a bit position (shown above), and superimpose it with other bit positions. Since only bit positions that are a “1” should be included, the variable \( R_k \) is introduced, with a value of “1” or “0”, corresponding to the data stream at the kth position of the repeating pattern or frame.

The Fourier transform pair of the data stream of length \( c \), can be expressed as:

\[
\sum_{k=1}^{c} R_k * \text{rect}\left(\frac{t - \frac{(2k-1)\tau}{2}}{\tau}\right) \leftrightarrow \sum_{k=1}^{c} R_k * \exp\left( -j\omega \frac{(2k-1)\tau}{2} \right) * \tau * \text{Sa}\left(\frac{\omega \tau}{2}\right).
\]

Since the code is periodic, with period \( \frac{c}{r} = T \), a Fourier series can be obtained from the transform using the property that \( F_n = \frac{1}{T} * F(\omega) \bigg|_{\omega = n\omega_0} \),

\[\omega_0 = \frac{2\pi}{T} = \frac{2\pi r}{c} \cdot \] In the above equation, \( n \) is the harmonic number of the series and \( F_n \) is the complex Fourier series representation for the harmonic such that: \( f(t) = \sum_{n=-\infty}^{\infty} F_n * \exp(jn\omega_0 t) \). \( f(t) \) is defined as the time domain signal consisting of integral multiples (>0) of the repeating data pattern. The above relationship is applicable since the signal has finite power and \( f\left(\frac{\pm T}{2}\right) = 0 \).

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Proof continued

Using the property of (2), and $F(\omega)$ from (1), the Fourier series of the data stream, $f(t)$, can be expressed as:

$$F_n = \frac{r}{c} \sum_{k=1}^{c} R_k \exp \left( -jn\frac{2\pi r}{c}(2k-1) \right) \cdot \frac{1}{r} \cdot \text{Sa} \left( \frac{n\frac{2\pi r}{c}1}{2} \right)$$

$$F_n = \frac{r}{c} \sum_{k=1}^{c} R_k \exp \left( -jn2\pi r(2k-1) \right) \cdot \frac{\sin \left( \frac{n\pi r}{rc} \right)}{n\pi} \cdot \frac{1}{r}$$

$$F_n = \frac{r}{c} \cdot \frac{1}{r} \cdot \frac{rc}{n\pi r} \cdot \sin \left( \frac{n\pi r}{rc} \right) \cdot \sum_{k=1}^{c} R_k \exp \left( -jn2\pi r(2k-1) \right)$$

$$F_n = \frac{n\pi}{r} \sum_{k=1}^{c} R_k \exp \left( -j\frac{n\pi(2k-1)}{c} \right)$$

To express $F_n$ in terms of real coefficients of sine and cosine, define $a_n$ and $b_n$ as:

$$a_n = 2 \text{ Re } \{F_n\} \text{ (cosine term)}$$
$$b_n = -2 \text{ Im } \{F_n\} \text{ (sine term)}, \text{ such that}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t).$$

Rewriting the expression for $F_n$, using Euler’s identity:

$$F_n = \sin \left( \frac{n\pi}{c} \right) \sum_{k=1}^{c} R_k \left[ \cos \left( \frac{n\pi(2k-1)}{c} \right) - j \sin \left( \frac{n\pi(2k-1)}{c} \right) \right]$$

$$a_n = \frac{n\pi}{c} \sum_{k=1}^{c} R_k \cos \left( \frac{n\pi(2k-1)}{c} \right)$$
$$b_n = \frac{n\pi}{c} \sum_{k=1}^{c} R_k \sin \left( \frac{n\pi(2k-1)}{c} \right)$$

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